Mathematics-II (Differential Equations)
Lecture Notes
April 18, 2020

by

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Laplace's Equation in spherical coordinates

If the given boundary problem involves spherical boundaries, we use Laplace's equation in spherical coordinates (r, θ, ϕ).

Example

Transform the Laplace's equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ into polar coordinates (r, θ, ϕ).

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Solution: If (x, y, z) be the Cartesian coordinate's of the point P whose spherical coordinates are (r, θ, ϕ), then

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi$$
$$and \qquad z = r\cos\theta \tag{1}$$

From (1)

$$r^2 = x^2 + y^2 + z^2, \qquad \tan\theta = \frac{(x^2+y^2)^{1/2}}{z}$$
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Laplace equation in ...

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$$\implies \frac{\partial u}{\partial x} = \sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}$$

$$\implies \frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial\theta}{\partial y}=\frac{\cos\theta\sin\phi}{r}, \qquad and \qquad \frac{\partial\theta}{\partial z}=-\frac{\sin\theta}{r} \tag{7}$$

And

$$\frac{\partial\phi}{\partial x}=-\frac{\sin\phi}{r\sin\theta}, \qquad and \qquad \frac{\partial\phi}{\partial y}=\frac{\cos\phi}{r\sin\theta} \qquad and \qquad \frac{\partial\phi}{\partial z}=0 \tag{8}$$

Now

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}+\frac{\partial u}{\partial\phi}\frac{\partial\phi}{\partial x}$$

$$\implies \frac{\partial u}{\partial x}=\sin\theta\cos\phi\frac{\partial u}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}$$

$$\implies \frac{\partial}{\partial x}=\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta \sin\phi}{r}, \qquad and \qquad \frac{\partial \theta}{\partial z} = -\frac{\sin\theta}{r} \tag{7}$$

And

$$\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}, \qquad and \qquad \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r\sin\theta} \qquad and \qquad \frac{\partial \phi}{\partial z} = 0 \tag{8}$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x}$$

$$\implies \frac{\partial u}{\partial x} = \sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial \phi}$$

$$\implies \frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

$$\frac{\partial\theta}{\partial y} = \frac{\cos\theta\sin\phi}{r}, \qquad and \qquad \frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r} \tag{7}$$

And

$$\frac{\partial\phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}, \qquad and \qquad \frac{\partial\phi}{\partial y} = \frac{\cos\phi}{r\sin\theta} \qquad and \qquad \frac{\partial\phi}{\partial z} = 0 \tag{8}$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial u}{\partial\phi}\frac{\partial\phi}{\partial x}$$

$$\implies \frac{\partial u}{\partial x} = \sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}$$

$$\implies \frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \times \left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\cos\phi\frac{\partial}{\partial r}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right) + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right) - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right)$$
$$\times\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial \phi}\right)$$

$$= \sin\theta\cos\phi\frac{\partial}{\partial r}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial \phi}\right)$$
$$+ \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial \phi}\right)$$
$$- \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial \phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\times\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\cos\phi\frac{\partial}{\partial r}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+ \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$- \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\times\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\cos\phi\frac{\partial}{\partial r}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+ \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$- \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\times\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\cos\phi\frac{\partial}{\partial r}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+ \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$- \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\times\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\cos\phi\frac{\partial}{\partial r}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+ \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$- \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$
$$\times\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\cos\phi\frac{\partial}{\partial r}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$
$$+ \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$
$$- \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\cos\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Laplace equation in ...

Dr. G.K. Prajapati

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Introduction

Thus

$$\frac{\partial^2 u}{\partial x^2} = \sin^2\theta\cos^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} + \frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} + \frac{\cos^\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial u}{\partial\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}. \quad (9)$$

Thus

$$\frac{\partial^2 u}{\partial x^2} = \sin^2\theta\cos^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} + \frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} + \frac{\cos^{\theta}\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial u}{\partial\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}. \quad (9)$$

Thus

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} = \sin^2\theta\cos^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} \\ -\frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} \\ +\frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial u}{\partial r} \\ +\frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} \\ +\frac{\cos^\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial u}{\partial\theta} \\ +\frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}.\end{aligned} \tag{9}$$

Thus

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \sin^2\theta\cos^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta}\\ &\quad -\frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi}\\ &\quad +\frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial u}{\partial r}\\ &\quad +\frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi}\\ &\quad +\frac{\cos^\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial u}{\partial\theta}\\ &\quad +\frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}. \qquad (9)\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \sin^2\theta\cos^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} \\
&\quad - \frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} \\
&\quad + \frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial u}{\partial r} \\
&\quad + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} \\
&\quad + \frac{\cos^\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial u}{\partial\theta} \\
&\quad + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}. \qquad (9)
\end{aligned}$$

Thus

$$\frac{\partial^2 u}{\partial x^2} = \sin^2\theta\cos^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} + \frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} + \frac{\cos^\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial u}{\partial\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}. \quad (9)$$

Thus

$$\frac{\partial^2 u}{\partial x^2} = \sin^2\theta\cos^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\cos^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta\cos^2\phi}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} + \frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\cos^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\cos^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} + \frac{\cos^\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\sin^2\phi}{r^2\sin\theta}\frac{\partial u}{\partial\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}. \tag{9}$$

Again

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$+\frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} \implies \sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial \phi}$$

$$\implies \frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Again

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$+\frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} \implies \sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial \phi}$$

$$\implies \frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Again

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$+\frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} \implies \sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial \phi}$$

$$\implies \frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Again

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$+\frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} \implies \sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial \phi}$$

$$\implies \frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Again

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$+\frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} \implies \sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial \phi}$$

$$\implies \frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Again

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$+\frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial y} \implies \sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial \phi}$$

$$\implies \frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Therefore

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$
$$\times\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\sin\phi\frac{\partial}{\partial r}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$
$$+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$
$$+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$
$$\times\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\sin\phi\frac{\partial}{\partial r}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$
$$+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$
$$+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\times\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\sin\phi\frac{\partial}{\partial r}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\times\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\sin\phi\frac{\partial}{\partial r}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\times\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\sin\phi\frac{\partial}{\partial r}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\times\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\sin\phi\frac{\partial}{\partial r}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Therefore

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$
$$\times\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

$$= \sin\theta\sin\phi\frac{\partial}{\partial r}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$
$$+\frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$
$$+\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\sin\phi\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial u}{\partial\theta} - \frac{\cos\phi}{r\sin\theta}\frac{\partial u}{\partial\phi}\right)$$

Thus

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\sin^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial u}{\partial\theta} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} - \frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} - \frac{\cos^\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\cos^2\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} - \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}. \quad (10)$$

Thus

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\sin^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta}$$

$$-\frac{2\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial u}{\partial\theta} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi}$$

$$-\frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

$$+\frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} - \frac{\cos^{\theta}\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r}\frac{\partial u}{\partial r}$$

$$+\frac{\cos\theta\cos^2\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$

$$-\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}. \quad (10)$$

Thus

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\sin^2\phi\frac{\partial^2 u}{\partial r^2} &+ \frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} \\
&-\frac{2\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial u}{\partial\theta} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} \\
&-\frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} \\
&+\frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} - \frac{\cos^{\theta}\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r}\frac{\partial u}{\partial r} \\
&+\frac{\cos\theta\cos^2\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \\
&-\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}. \quad (10)
\end{aligned}
$$

Thus

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\sin^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta}$$
$$-\frac{2\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial u}{\partial\theta} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi}$$
$$-\frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$
$$+\frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} - \frac{\cos^\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r}\frac{\partial u}{\partial r}$$
$$+\frac{\cos\theta\cos^2\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$
$$-\frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}.(10)$$

Thus

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\sin^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial u}{\partial\theta} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} - \frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} - \frac{\cos^\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\cos^2\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} - \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}. \quad (10)$$

Thus

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \sin^2\theta\sin^2\phi\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial^2 u}{\partial r\partial\theta} \\
&\quad - \frac{2\sin\theta\cos\theta\sin^2\phi}{r^2}\frac{\partial u}{\partial\theta} + \frac{2\sin\phi\cos\phi}{r}\frac{\partial^2 u}{\partial r\partial\phi} \\
&\quad - \frac{\sin\phi\cos\phi}{r^2}\frac{\partial u}{\partial\phi} + \frac{\cos^2\theta\sin^2\phi}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta\sin^2\phi}{r^2}\frac{\partial^2 u}{\partial\theta^2} \\
&\quad + \frac{2\cos\theta\sin\phi\cos\phi}{r^2\sin\theta}\frac{\partial^2 u}{\partial\theta\partial\phi} - \frac{\cos^{\theta}\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r}\frac{\partial u}{\partial r} \\
&\quad + \frac{\cos\theta\cos^2\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi} + \frac{\cos^2\phi}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} \\
&\quad - \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\frac{\partial u}{\partial\phi}. \quad (10)
\end{aligned}$$

Finally

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial z} \implies \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\implies \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Therefore

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

Laplace equation in ...

Dr. G.K. Prajapati

LNJPIT, Chapra

Introduction

Finally

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial z} \implies \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\implies \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Therefore

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

Finally

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial z} \implies \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\implies \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Therefore

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

Finally

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial z} \implies \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\implies \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Therefore

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

Finally

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial z} \implies \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\implies \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Therefore

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

Finally

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial z} \implies \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\implies \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Therefore

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

Finally

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial z} \implies \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\implies \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

Therefore

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

Finally

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial z} \implies \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

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Thus

$$\frac{\partial^2 u}{\partial z^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \quad (11)$$

Adding (9), (10) and (11)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}.$$

Hence Laplace equation in spherical coordinates is

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

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Laplace equation in ...

Dr. G.K. Prajapati

LNJPIT, Chapra

Introduction

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Thanks !!!