Mathematics-II (Complex Variable)
Lecture Notes
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by

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Definition

Any function which satisfies the Laplaces equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

is known as a harmonic function.

Theorem

If $f(z) = u + iv$ is an analytic function, then u and v are both harmonic functions. Such functions u and v are called **Conjugate harmonic functions** *if $u + iv$ is also analytic function.*

Example

Prove that $u = x^2 - y^2$ and $v = \dfrac{y}{x^2 + y^2}$ are harmonic functions of (x, y), but are not harmonic conjugates.

Solution:

$$\frac{\partial u}{\partial x} = 2x \, , \frac{\partial^2 u}{\partial x^2} = 2, \; \frac{\partial u}{\partial y} = -2y, \; \frac{\partial^2 u}{\partial y^2} = -2$$

Thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

$u(x, y)$ satisfies Laplace equation, hence $u(x, y)$ is harmonic.

Now

$$\frac{\partial v}{\partial x} = \frac{-2xy}{(x^2+y^2)^2},$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)^2(-2y) - (-2xy)2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-2y) - (-2xy)2(2x)}{(x^2+y^2)^3} = \frac{(6x^2y - 2y^3)}{(x^2+y^2)^3}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2).1 - y.2(x^2+y^2)(2y)}{(x^2+y^2)^2} = \frac{(x^2-y^2)}{(x^2+y^2)^2},$$

$$= \frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(-2y) - (x^2-y^2)2(x^2+y^2)(2y)}{(x^2+y^2)^4} =$$

$$\frac{(x^2+y^2)(-2y) - (x^2-y^2)2(2y)}{(x^2+y^2)^3} = \frac{(-6x^2y + 2y^3)}{(x^2+y^2)^3}$$

Thus

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\left(\frac{(6x^2y-2y^3)}{(x^2+y^2)^3}\right)+\left(\frac{(-6x^2y+2y^3)}{(x^2+y^2)^3}\right)=0.$$

$v(x,y)$ satisfies Laplace equation, hence $v(x,y)$ is harmonic.
But

$$\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}\qquad and \qquad \frac{\partial u}{\partial y}\neq-\frac{\partial v}{\partial x}.$$

Therefore u and v are not harmonic conjugates.

METHOD TO FIND THE CONJUGATE FUNCTION:**Case I.** Given. If $f(z) = u + iv$, and u is known.
Claim: We have to find conjugation function v.

Example

Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find a function v such that $f(z) = u + iv$ is analytic. Also express $f(z)$ in terms of z.

Solution: We have, $u = x^2 - y^2 - 2xy - 2x + 3y$ so that

$$\frac{\partial u}{\partial x} = 2x - 2y - 2 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = -2y - 2x + 3 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

Thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

$u(x, y)$ satisfies Laplace equation, hence $u(x, y)$ is harmonic.
We know that

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Using $C - R$ equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$,

qquad and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

Putting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we get

$$dv = -(-2y - 2x + 3)dx + (2x - 2y - 2)dy$$

The R.H.S. is an exact differential equation of the form $Mdx + Ndy$. Hence its solution is

$$v = -\int(-2y - 2x + 3)dx + \int(-2y - 2)dy \implies v = 2xy + x^2 - 3x - y^2 - 2y + c$$

Now,

$$
\begin{aligned}
f(z) &= u + iv \\
&= (x^2 - y^2 - 2xy - 2x + 3y) + i(2xy + x^2 - 3x - y^2 - 2 \\
&= (x^2 - y^2 + 2ixy) + (ix^2 - iy^2 - 2xy) - (2 + 3i)x - i(2 \\
&= (x^2 - y^2 + 2ixy) + i(x^2 - y^2 + 2ixy) - (2 + 3i)x - i(2 \\
&= (x + iy)^2 + i(x + iy)^2 - (2 + 3i)(x + iy) + ic \\
&= z^2 + iz^2 - (2 + 3i)z + ic \\
&= (1 + i)z^2 - (2 + 3i)z + ic
\end{aligned}
$$

Which is the required expression of f (z) in terms of z.

Thanks !!!