Mathematics-II (Complex Variable)
Lecture Notes
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by

Dr. G.K.Prajapati
Department of Applied Science and Humanities
LNJPIT, Chapra, Bihar-841302

Let $x_0, x_1, x_2, ..., x_n$ be the equally spaced data and h be the step length in the given data. Then

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f(x_0)}{1!h} + (x - x_0)(x - x_1)\frac{\Delta^2 f(x_0)}{2!h^2} + (x - x_0)(x - x_1)(x - x_2)\frac{\Delta^3 f(x_0)}{3!h^3} + ... + (x - x_0)(x - x_1)(x - x_2)...(x - x_{n-1})\frac{\Delta^n f(x_0)}{n!h^n}.$$

is called the Newton's forward difference interpolation formula.

Example

For the data construct the forward difference formula. Hence, find $f(0.5)$.

x	-2	-1	0	1	2	3
$f(x)$	15	5	1	3	11	25

Solution: We have the following difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
-2	15					
		-10				
-1	5		6			
		-4		0		
0	1		6		0	
		2		0		0
1	3		6		0	
		8		0		
2	11		6			
		14				
3	25					

From the table, we conclude that the data represents a quadratic polynomial. We have $h = 1$. The Newton's forward difference formula is given by

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f(x_0)}{1!h} + (x - x_0)(x - x_1)\frac{\Delta^2 f(x_0)}{2!h^2} + (x-x_0)(x-x_1)(x-x_2)\frac{\Delta^3 f(x_0)}{3!h^3} + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\frac{\Delta^4 f(x_0)}{4!h^4} + (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)\frac{\Delta^5 f(x_0)}{5!h^5}.$$

By putting the required values from the table we have,

$$f(x) = 15 + (x+2)\frac{(-10)}{1!.1} + (x+2)(x+1)\frac{6}{2!1^2} + (x+2)(x+1)(x-0)\frac{0}{3!1^3} + (x+2)(x+1)(x-0)(x-1)\frac{0}{4!1^4} + (x+2)(x+1)(x-0)(x-1)(x-2)\frac{0}{5!1^5}.$$

$$f(x) = 15 + (x+2)(-10) + (x+2)(x+1)(3) = 15 - 10x - 20 + 3x^2 + 9x + 6 = 3x^2 - x + 1.$$

We obtain $f(0.5) = 3(0.5)^2 - 0.5 + 1 = 0.75 - 0.5 + 1 = 1.25.$

Example

A third degree polynomial passes through the points $(0,-1),(1,1),(2,1) and (3,-2)$. Determine this polynomial using Newton's forward interpolation formula. Hence, find the value at 1.5.

Solution: We have the following difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	-1	2		
1	1	0	-2	-1
2	1	-3	-3	
3	-2			

Table: Forward Difference Table

From the table, we conclude that the data represents a cubic polynomial. We have $h = 1$. The Newton's forward difference formula is given by

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f(x_0)}{1!h} + (x - x_0)(x - x_1)\frac{\Delta^2 f(x_0)}{2!h^2} + (x - x_0)(x - x_1)(x - x_2)\frac{\Delta^3 f(x_0)}{3!h^3}.$$

By putting the required values from the table we have,

$$f(x) = (-1)+(x-0)\frac{2}{1!.1}+(x-0)(x-1)\frac{-2}{2!1^2}+(x-0)(x-1)(x-2)\frac{-1}{3!1^3}.$$

$$f(x) = -1 + 2x - x(x-1) - \frac{1}{6}x(x-1)(x-2) = -1 + 2x - x^2 + x - \frac{1}{6}(x^3 - 3x^2 + 2x)$$

$$f(x) = -1 + (2 - 2/6 + 1)x - (1 - 3/6)x^2 - 1/6x^3 = -1 + (8/3)x - (1/2)x^2 - (1/6)x^3$$

We obtain
$f(1.5) = -1 + (8/3)(1.5) - (1/2)(1.5)^2 - (1/6)(1.5)^3 = 1.3125.$

Let $x_0, x_1, x_2, ..., x_n$ be the equally spaced data and h be the step length in the given data. Then

$$f(x) = f(x_n) + (x - x_n)\frac{\nabla f(x_n)}{1!h} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 f(x_n)}{2!h^2} + (x - x_n)(x - x_{n-1})(x - x_{n-2})\frac{\nabla^3 f(x_n)}{3!h^3} + \ldots + (x - x_n)(x - x_{n-1})(x - x_{n-2})\ldots(x - x_1)\frac{\nabla^n f(x_n)}{n!h^n}.$$

is called the Newton's backward difference interpolation formula.

Example

Using Newton's backward difference interpolation, interpolate at $x = 1.0$ from the following data.

x	0.1	0.3	0.5	0.7	0.9	1.1
$f(x)$	-1.699	-1.073	-0.375	0.443	1.429	2.631

Solution: We have the following difference table:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$
0.1	−1.699					
		0.626				
0.3	−1.073		0.072			
		0.698		0.048		
0.5	−0.375		0.120		0	
		0.818		0.048		0
0.7	0.443		0.168		0	
		0.986		0.048		
0.9	1.429		0.216			
		1.202				
1.1	2.631					

From the table, We have $h = 0.2$. The Newton's backward difference formula is given by

$$f(x) = f(x_n) + (x - x_n)\frac{\nabla f(x_n)}{1!h} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 f(x_n)}{2!h^2} + (x - x_n)(x - x_{n-1})(x - x_{n-2})\frac{\nabla^3 f(x_n)}{3!h^3}.$$

By putting the required values from the table we have,

$$f(x) = 2.631 + (x-1.1)\frac{1.202}{1!(0.2)} + (x-1.1)(x-0.9)\frac{0.216}{2!(0.2)^2} + (x-1.1)(x-0.9)(x-0.7)\frac{0.048}{3!(0.2)^3}.$$

$$f(x) = 2.631 + 6.01(x-1.1) + 2.7(x-1.1)(x-0.9) + (x-1.1)(x-0.9)(x-0.7).$$

Since, we have not been asked to find the interpolation polynomial, we may not simplify this expression. At $x = 1.0$, we obtain

$f(1.0) = 2.631 + 6.01(1.0-1.1) + 2.7(1.0-1.1)(1.0-0.9) + (1.0-1.1)(1.0-0.9)(1.0-0.7)$
$= 2.631 + 6.01(-0.1) + 2.7(-0.1)(0.1) + (-0.1)(0.1)(-0.3) = 2.004.$

Thanks !!!